

Two-step spacetime deformation induced dynamical torsion

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We extend the geometrical ideas of the spacetime deformations to study the physical foundation of the post-Riemannian geometry. To this aim, we construct the theory of *two-step spacetime deformation* as a guiding principle. We address the theory of teleparallel gravity and construct a consistent Einstein-Cartan (EC) theory with the *dynamical torsion*. We show that the equations of the standard EC theory, in which the equation defining torsion is the algebraic type and, in fact, no propagation of torsion is allowed, can be equivalently replaced by the set of *modified EC equations* in which the torsion, in general, is *dynamical*. The special physical constraint imposed upon the spacetime deformations yields the short-range propagating spin-spin interaction.

Keywords: Spacetime deformations, Post-Riemannian geometry, Dynamical torsion

I. INTRODUCTION

At present, the papers on the gauge treatment of gravity are the considerable part of all gravitational investigations. In its present formulation, this exploits the language of the fundamental geometric structure known as a fiber bundle, which provides a unified picture of gravity modified models based on several Lie groups, see e.g. [1]-[12]. These efforts mainly focus on the physical foundation of torsion, and its connection to quantum gravity and microphysics. The Einstein-Cartan (EC) theory is the minimal extension of the general relativity, which considers curvature and torsion as representing independent degrees of freedom, and relating torsion to the density of intrinsic angular momentum. In the standard EC theory, the equation defining torsion is of algebraic type, and not a differential equation, and that no propagation of torsion is allowed. As is known from the weak interaction, the causality reasons do not respect a contact interaction. Therefore, many modifications of the EC theory have been proposed in recent years, see e.g. [9], right up to introducing a higher order gravitational Lagrangian of quadratic models [13], different from the simple scalar curvature. This aimed to obtain a differential field equation for the torsion tensor, instead of an algebraic one. Even though a strong emphasis has been placed on this issue throughout the development of modern physics, all these approaches are subject to many uncertainties. On the other hand, a general way to deform the spacetime metric with constant curvature has been explicitly posed by [14]-[16]. The problem was initially solved only for 3D spaces, but consequently it was solved also for spacetimes of any dimension. It was proved that any semi-Riemannian metric can be obtained as a deformation of constant curvature metric, this deformation being parameterized by a 2-form. These results are fully recovered and generalized by [10], where a novel definition of spacetime metric deformations, parameterized in terms of scalar field matrices, is proposed.

In this article we construct the theory of *two-step spacetime deformation* (TSSD), which generalizes and, in particular cases, fully reproduces the results of the conventional theory of spacetime deformation. We show that through a non-trivial choice of explicit form of a *deformation* tensor, we have a way to derive different post Riemannian spacetime structures such as: 1) the Weitzenböck space, W_4 , underlying a teleparallelism theory of gravity, see e.g. [17]; 2) the RC manifold, U_4 , underlying Einstein-Cartan theory also called Einstein-Cartan-Sciama-Kibble, for a comprehensive references, see for example [8, 9, 12]); 3) or even the most general linear connection of MAG theory taking values in the Lie-algebra of the 4D-affine group, $A(4, R) = R^4 \ltimes GL(4, R)$ (the semi-direct product of the group of 4D-translations and general linear 4D-transformations), see e.g. [6, 7]. We are mainly interested in the formulation of physical aspects of the EC theory from a novel, TSSD, view point. In the framework of the TSSD- U_4 theory, we address the key problem of the *dynamical torsion*. We show that the equations of the standard EC theory can be equivalently replaced by the set of *modified EC equations* in which the torsion, in general, is a *dynamical*. The physical constraint will be imposed upon the spacetime deformations, which yields the short-range propagating spin-spin interaction. We will proceed according to the following structure. In section 2 we construct the TSSD theory as the guiding principle. In section 3, in case of particular spacetime deformations, we obtain the theory of teleparallel gravity. In section 4, emerging structures are embedded into the foundations of the TSSD- U_4 theory in both a tensorial form and a language of the differential forms, and the equation of the short-range propagating torsion is derived. The concluding remarks are presented in section 5. We use the Greek alphabet ($\mu, \nu, \rho, \dots = 0, 1, 2, 3$) to denote the holonomic world indices related to curved spacetime \mathcal{M}_4 , and the Latin alphabet ($a, b, c, \dots = 0, 1, 2, 3$) to denote the anholonomic indices related to the tangent space.

II. THE TSSD AS A GUIDING PRINCIPLE

When considering several connections with different curvature and torsion, one takes spacetime simply as a manifold, and connections as additional structures, see e.g. [21, 22]. From this view point, below we shall tackle the problem of spacetime deformation. To start with, let us consider the holonomic metric defined in the Riemann space, V_4 , as

$$\check{g} = \check{g}_{\mu\nu} \check{\vartheta}^\mu \otimes \check{\vartheta}^\nu = \check{g}(\check{e}_\mu, \check{e}_\nu) \check{\vartheta}^\mu \otimes \check{\vartheta}^\nu, \quad (1)$$

with components, $\check{g}_{\mu\nu} = \check{g}(\check{e}_\mu, \check{e}_\nu)$ in the dual holonomic base $\{\check{\vartheta}^\mu \equiv d\check{x}^\mu\}$. All magnitudes related to the Riemann space, V_4 , will be denoted with an over $\check{\cdot}$. The space, V_4 , has at each point a tangent space, $\check{T}_x V_4$, spanned by the four tetrad fields, $\check{e}_a = \check{e}_a^\mu \check{\partial}_\mu$, which relate \check{g} to the tangent space metric, $o_{ab} = \text{diag}(+---)$, by

$$o_{ab} = \check{g}(\check{e}_a, \check{e}_b) = \check{g}_{\mu\nu} \check{e}_a^\mu \check{e}_b^\nu. \quad (2)$$

The coframe members are $\check{\vartheta}^b = \check{e}_\mu^b d\check{x}^\mu$, such that $\check{e}_a \rfloor \check{\vartheta}^b = \delta_a^b$, where \rfloor denoting the interior product, namely, this is a C^∞ -bilinear map $\rfloor : \Omega^1 \rightarrow \Omega^0$ with Ω^p denotes the C^∞ -module of differential p-forms on V_4 . In components $\check{e}_a^\mu \check{e}_\mu^b = \delta_a^b$. One can consider general transformations of the linear group, $GL(4, R)$, taking any base into any other set of four linearly independent fields. The notation, $\{\check{e}_a, \check{\vartheta}^b\}$, will be used below for general linear frames. Relation (2) has the converse $\check{g}_{\mu\nu} = o_{ab} \check{e}_\mu^a \check{e}_\nu^b$ because $\check{e}_a^\mu \check{e}_\nu^a = \delta_\nu^\mu$. The anholonomy objects read

$$\check{C}^a : = d\check{\vartheta}^a = \frac{1}{2} \check{C}^a_{bc} \check{\vartheta}^b \wedge \check{\vartheta}^c, \quad (3)$$

where the anholonomy coefficients, \check{C}^a_{bc} , which represent the curls of the base members are

$$\check{C}^c_{ab} = -\check{\vartheta}^c([\check{e}_a, \check{e}_b]) = \check{e}_a^\mu \check{e}_b^\nu (\check{\partial}_\mu \check{e}_\nu^c - \check{\partial}_\nu \check{e}_\mu^c) = -\check{e}_\mu^c [\check{e}_a(\check{e}_b^\mu) - \check{e}_b(\check{e}_a^\mu)]. \quad (4)$$

The (anholonomic) Levi-Civita (or Christoffel) connection can be written as

$$\check{\Gamma}_{ab} : = \check{e}_{[a} \rfloor d\check{\vartheta}_{b]} - \frac{1}{2} (\check{e}_a \rfloor \check{e}_b \rfloor d\check{\vartheta}_c) \wedge \check{\vartheta}^c \quad (5)$$

where $\check{\vartheta}_c$ is understood as the down indexed 1-form $\check{\vartheta}_c = o_{cb} \check{\vartheta}^b$.

A. Model building: spacetime deformation

Next, we write the norm, ds , of the infinitesimal displacement, $d x^\mu$, on the general smooth differential 4D-manifold \mathcal{M}_4 , in terms of the spacetime structures of V_4 , as

$$ds = \Omega_\mu^\nu \check{e}_\nu \check{\vartheta}^\mu = \Omega_b^a \check{e}_a \check{\vartheta}^b = e_\rho \vartheta^\rho = e_a \vartheta^a \in \mathcal{M}_4, \quad (6)$$

where Ω_μ^ν is the world-deformation tensor, $\{e_a = e_a^\rho e_\rho\}$ is the frame and $\{\vartheta^a = e_a^\rho \vartheta^\rho\}$ is the coframe defined on \mathcal{M}_4 , such that $e_a \rfloor \vartheta^b = \delta_a^b$, or in components, $e_a^\mu e_\mu^b = \delta_a^b$, also the procedure can be inverted $e_a^\rho e_\rho^a = \delta_\rho^a$, provided

$$\Omega_\mu^\nu = \pi_\mu^\rho \pi_\rho^\nu, \quad \Omega_b^a = \pi_c^a \pi_b^c = \Omega_\mu^\nu \check{e}_\nu^a \check{e}_b^\mu, \quad e_\rho = \pi_\rho^\nu \check{e}_\nu \equiv \partial_\rho, \quad \vartheta^\rho = \pi_\mu^\rho \check{\vartheta}^\mu \equiv d x^\rho, \quad x^\rho \in \mathcal{U} \in \mathcal{M}_4. \quad (7)$$

Hence the deformation tensor, Ω_b^a , yields local tetrad deformations

$$e_a \vartheta^a = \Omega_b^a \check{e}_a \check{\vartheta}^b, \quad e_c = \pi_c^a \check{e}_a, \quad \vartheta^c = \pi_c^b \check{\vartheta}^b. \quad (8)$$

A general spin connection then transforms according

$$\omega_{b\mu}^a = \pi_c^a \check{\omega}_{d\mu}^c \pi_b^d + \pi_c^a \partial_\mu \pi_b^c. \quad (9)$$

Deformations (8) restore a formalism of the spacetime metric deformation proposed in [10]. Therefore, following this work, the matrices, $\pi(x) : = (\pi_b^a)(x)$, can be called *first deformation matrices*, and the matrices

$$\gamma_{cd}(x) = o_{ab} \pi_c^a(x) \pi_d^b(x), \quad (10)$$

second deformation matrices. The matrices, $\pi_c^a(x) \in GL(4, R) \forall x$, in general, give rise to the right cosets of the Lorentz group, i.e. they are the elements of the quotient group $GL(4, R)/SO(3, 1)$. If we deform the tetrad according to (8), in general, we have two choices to recast metric as follows: either writing the deformation of the metric in the space of tetrads or deforming the tetrad field:

$$g = o_{ab} \pi_c^a \pi_d^b \check{\vartheta}^c \otimes \check{\vartheta}^d = \gamma_{cd} \check{\vartheta}^c \otimes \check{\vartheta}^d = o_{ab} \vartheta^a \otimes \vartheta^b. \quad (11)$$

In the first case, the contribution of the Christoffel symbols, constructed by the metric γ_{ab} , reads

$$\Gamma_{bc}^a = \frac{1}{2} \left(\check{C}_{bc}^a - \gamma^{aa'} \gamma_{bb'} \check{C}_{a'c}^{b'} - \gamma^{aa'} \gamma_{cc'} \check{C}_{a'b}^{c'} \right) + \frac{1}{2} \gamma^{aa'} (\check{e}_c \rfloor d \gamma_{ba'} - \check{e}_b \rfloor d \gamma_{ca'} - \check{e}_{a'} \rfloor d \gamma_{bc}). \quad (12)$$

The second deformation matrix, γ_{ab} , can be decomposed in terms of symmetric, $\pi_{(ab)}$, and antisymmetric, $\pi_{[ab]}$, parts of the matrix $\pi_{ab} = o_{ac} \pi_b^c$ as

$$\gamma_{ab} = \Upsilon^2 o_{ab} + 2\Upsilon \Theta_{ab} + o_{cd} \Theta_a^c \Theta_b^d + o_{cd} (\Theta_a^c \varphi_b^d + \varphi_a^c \Theta_b^d) + o_{cd} \varphi_a^c \varphi_b^d, \quad (13)$$

where

$$\pi_{ab} = \Upsilon o_{ab} + \Theta_{ab} + \varphi_{ab} \quad (14)$$

$\Upsilon = \pi_a^a$, Θ_{ab} is the traceless symmetric part and φ_{ab} is the skew symmetric part of the first deformation matrix. Consequently, the deformed metric, can be split as

$$g_{\mu\nu}(\pi) = \Upsilon^2(\pi) \check{g}_{\mu\nu} + \gamma_{\mu\nu}(\pi), \quad (15)$$

where

$$\gamma_{\mu\nu}(\pi) = [\gamma_{ab} - \Upsilon^2(\pi) o_{ab}] \check{e}_\mu^a \check{e}_\nu^b. \quad (16)$$

The inverse deformed metric reads

$$g^{\mu\nu}(\pi) = o^{cd} \pi^{-1a}_c \pi^{-1b}_d \check{e}_a^\mu \check{e}_b^\nu, \quad (17)$$

where $\pi^{-1a}_c \pi_b^c = \pi_b^c \pi^{-1a}_c = \delta_b^a$. In the second case, let us write the commutation table for the anholonomic frame, $\{e_a\}$,

$$[e_a, e_b] = -\frac{1}{2} C_{ab}^c e_c, \quad (18)$$

and define a dual expression of the new anholonomy objects, C_{bc}^a ,

$$C^a : = d\vartheta^a = \frac{1}{2} C_{bc}^a \vartheta^b \wedge \vartheta^c = \frac{1}{2} (\partial_\mu e_\nu^c - \partial_\nu e_\mu^c) dx^\mu \wedge dx^\nu, \quad (19)$$

where

$$C_{bc}^a = \pi_e^a \pi^{-1d}_b \pi^{-1f}_c \check{C}_{df}^e + 2 \pi_f^a \check{e}_g^\mu \left(\pi^{-1g}_{[b} \partial_\mu \pi^{-1f}_{c]} \right). \quad (20)$$

In the particular case of constant metric in tetradic space, the deformed connection can be written as

$$\Gamma_{bc}^a = \frac{1}{2} \left(C_{bc}^a - o^{aa'} o_{bb'} C_{a'c}^{b'} - o^{aa'} o_{cc'} C_{a'b}^{c'} \right). \quad (21)$$

The usual Levi-Civita connection corresponding to the metric (11) is related to the original connection by the relation

$$\Gamma_{\rho\sigma}^\mu = \check{\Gamma}_{\rho\sigma}^\mu + \Pi_{\rho\sigma}^\mu, \quad (22)$$

provided

$$\Pi_{\rho\sigma}^\mu = 2g^{\mu\nu} \check{g}_{\nu(\rho} \nabla_{\sigma)} \Upsilon - \check{g}_{\rho\sigma} g^{\mu\nu} \nabla_\nu \Upsilon + \frac{1}{2} g^{\mu\nu} (\nabla_\rho \gamma_{\nu\sigma} + \nabla_\sigma \gamma_{\rho\nu} - \nabla_\nu \gamma_{\rho\sigma}), \quad (23)$$

where the contravariant deformed metric, $g^{\nu\rho}$, is defined as the inverse of $g_{\mu\nu}$, such that $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$. Hence, the connection deformation $\Pi_{\rho\sigma}^\mu$ acts like a force that deviates the test particles from the geodesic motion in the space, V_4 .

B. The post-Riemannian geometry

We now assume that a deformation $(\check{e}, \check{\vartheta}) \rightarrow (e, \vartheta)$ is performed, according to the following heuristic map, in *two-steps*:

$$\begin{array}{ccc}
 & (\check{e}(\check{x}), \check{\vartheta}(\check{x})) & \\
 \pi(x) \swarrow & \nearrow \pi(\check{x}) & \\
 & (e(\check{x}), \vartheta(\check{x})) & \\
 \sigma(x) \searrow & \nearrow & \\
 & (e(x), \vartheta(x)) &
 \end{array}$$

Two-step deformation map

provided, we require that the first deformation matrix, $\dot{\pi}(\check{x}) : = (\dot{\pi}_b^a)(\check{x})$, satisfies the following peculiar condition:

$$\dot{\pi}_c^a(\check{x}) \partial_\mu \pi^{-1}{}_b^c(\check{x}) = \check{\omega}_{b\mu}^a(\check{x}), \quad (24)$$

where $\check{\omega}_{b\mu}^a(\check{x})$ is the spin connection defined in the Riemann space. Whereas,

$$\begin{aligned}
 \dot{\Omega}_\mu{}^\nu &= \dot{\pi}_\mu{}^\rho \dot{\pi}_\rho{}^\nu, & \dot{\Omega}_b{}^a &= \dot{\pi}_c^a \dot{\pi}_b^c = \dot{\Omega}_\mu{}^\nu \check{e}_\nu^a \check{e}_b^\mu, \\
 \dot{e}_\rho &= \dot{\pi}_\rho{}^\nu \check{e}_\nu \equiv \partial_\rho = \frac{\partial}{\partial x^\rho}, & \dot{\vartheta}^\rho &= \dot{\pi}_\mu{}^\rho \check{\vartheta}^\mu \equiv d\check{x}^\rho.
 \end{aligned} \quad (25)$$

Under a local spacetime deformation $\dot{\pi}(\check{x})$, the tetrad changes according to

$$\dot{e}_a \dot{\vartheta}^a = \dot{\Omega}^a{}_b \check{e}_a \check{\vartheta}^b, \quad \dot{e}_c = \dot{\pi}_c^a \check{e}_a, \quad \dot{\vartheta}^c = \dot{\pi}_b^c \check{\vartheta}^b. \quad (26)$$

Since we are interested only in a peculiar condition (24) to be held, then it is completely satisfactory for further consideration to write the first deformation matrix, $\dot{\pi}(\check{x})$, in the form of a particular solution to (24). To derive this solution, we recall that for an arbitrary matrix M ([19]),

$$tr \{ M^{-1} \partial_\mu M \} = \partial_\mu \ln |M|, \quad (27)$$

where $|...|$ denotes the determinant, tr the trace. According to it, in matrix notation $\check{\omega}_\mu := (\check{\omega}_{b\mu}^a)$, the equation (24) becomes

$$tr \left\{ \dot{\pi}(\check{x}) \partial_\mu \pi^{-1}(\check{x}) \right\} = - \partial_\mu \ln | \dot{\pi}(\check{x}) | = tr \check{\omega}_\mu(\check{x}), \quad (28)$$

which gives

$$| \dot{\pi}(\check{x}) | = | \dot{\pi}(0) | \exp \left\{ - \int_0^{\check{x}} tr \check{\omega}_\mu(\check{x}) d\check{x}'^\mu \right\}. \quad (29)$$

A particular solution to (24) is then

$$\dot{\pi}(\check{x}) = \dot{\pi}(0) \exp \left[- \int_0^{\check{x}} \check{\omega}_\mu(\check{x}) d\check{x}'^\mu \right]. \quad (30)$$

This is not generally the case. However, a general solution can be obtained by replacing $\dot{\pi}(0) \rightarrow \pi_B(\check{x}) \equiv \dot{\pi}(0) B(\check{x})$ in the expression (30), where $B(\check{x})$ is any proper matrix: $|B(\check{x})| = 1$. Before we report on the further key points of physical foundation of post-Riemannian geometry that have been used, for the benefit of the reader, we turn back to discussion of the geometrical implications of equation (30) which resembles the exponential of a bivector. Recall that the bivectors are quantities from geometric algebra, clifford algebra and the exterior algebra, which are generated by the exterior product on vectors. They are used to generate rotations in any dimension through the exponential map,

and are a useful tool for classifying such rotations, see e.g. [18]. All bivectors in four dimensions can be generated using at most two exterior products and four vectors. In the case of spacetime rotations, the geometric algebra is $Cl_{3,1}(R)$, and the subspace of bivectors is $\wedge^2 R_{3,1}$. Accordingly, the exponential map (30) generates set of all arbitrary rotations (26) of the orthonormal frame $\check{e}_a(\check{x})$ in tangent space, which form the Lorentz group. On the other hand, the universality of gravitation allows the Levi-Civita connection to be interpreted as part of the spacetime definition. The form of the Riemannian connection (5), which is a function of tetrad fields and their derivatives, shows that the relative orientation of the orthonormal frame $\check{e}_a(\check{x} + d\check{x})$ with respect to $\check{e}_a(\check{x})$ (parallel transported to $(\check{x} + d\check{x})$) is completely fixed by the metric. Since a change in this orientation is described by Lorentz transformations, it does not induce any gravitational effects; therefore, from the point of view of the principle of equivalence, there is no reason to prevent independent (due to arbitrary deformations (26)) Lorentz rotations of local frames in the space under consideration. If we want to use this freedom, the spin connection should contain a part which is independent of the metric, which will realize an independent Lorentz rotation of frames under parallel transport. In this way, we are led to a description of gravity which is not in Riemann space. If all inertial frames at a given point are treated on an equal footing, the spacetime has to have torsion, which is the antisymmetric part of the affine connection. The concept of a linear connection as an independent and primary structure of spacetime is the fundamental proposal put forward by Élie Cartan's geometrical analysis [20]. Another remark on the form of more generic spacetime deformation, $\pi(x)$, not subject to the condition (24), is also in order to validate our peculiar choice: when torsion is nonvanishing, the affine connection is no longer coincident with the Levi-Civita connection, and the geometry is no longer Riemannian, but one has a Riemann-Cartan U_4 spacetime, with a nonsymmetric, but metric-compatible, connection. Teleparallel gravity, in turn, represented a new way of including torsion into general relativity, an alternative to the scheme provided by the usual Einstein-Cartan-Sciama-Kibble approach. However, for a specific choice of free parameters, teleparallel gravity shows up as a theory completely equivalent to Einstein's general relativity, in which case it is usually referred to as the teleparallel equivalent of general relativity. From this point of view, curvature and torsion are simply alternative ways of describing the gravitational field, and consequently related to the same degrees of freedom of gravity. The fundamental difference between these two theories above was that, whereas in the former torsion is a propagating field, in the latter it is not, a point which can be considered a drawback of this model. Therefore, we have to separate, from the very outset, these two different cases. This motivates our choice of a double deformation map, with the peculiar condition (24). Namely, we deal with the spacetime deformation $\pi(x)$, to be consisted of two ingredient deformations $(\overset{\bullet}{\pi}(\check{x}), \sigma(x))$. By virtue of (24) or (30), the general deformed spin connection vanishes:

$$\overset{\bullet}{\omega}^a_{b\mu} = \overset{\bullet}{\pi}^a_c \overset{\bullet}{\omega}^c_{d\mu} \overset{\bullet}{\pi}^d_b + \overset{\bullet}{\pi}^a_c \overset{\bullet}{\partial}_\mu \overset{\bullet}{\pi}^c_b = \overset{\bullet}{e}^a_\sigma \overset{\bullet}{\Gamma}^\sigma_{\rho\mu} \overset{\bullet}{e}^\rho_b + \overset{\bullet}{e}^a_\rho \overset{\bullet}{\partial}_\mu \overset{\bullet}{e}^\rho_b \equiv 0. \quad (31)$$

In fact, a general linear connection, $\overset{\bullet}{\Gamma}^\mu_{\rho\sigma}$, is related to the corresponding spin connection, $\overset{\bullet}{\omega}^a_{b\mu}$, through the inverse

$$\overset{\bullet}{\Gamma}^\mu_{\rho\sigma} = \overset{\bullet}{e}^a_\mu \overset{\bullet}{\partial}_\sigma \overset{\bullet}{e}^a_\rho + \overset{\bullet}{e}^a_\mu \overset{\bullet}{\omega}^a_{b\sigma} \overset{\bullet}{e}^b_\rho = \overset{\bullet}{e}^a_\mu \overset{\bullet}{\partial}_\sigma \overset{\bullet}{e}^a_\rho, \quad (32)$$

which is the the Weitzenböck connection revealing the Weitzenböck spacetime W_4 of the teleparallel gravity (see next the section). Thus, $\overset{\bullet}{\pi}(\check{x})$ can be referred to as the Weitzenböck deformation matrix. The Weitzenböck connection is a connection presenting a non-vanishing torsion, but a vanishing curvature. This recovers a particular case of the teleparallel gravity theory with the dynamical torsion. All magnitudes related with the teleparallel gravity will be denoted with an over ' \bullet '. Furthermore, we will be able to generalize the EC equations for which the spin generates a dynamical torsion part (section 4), associated with spacetime deformation $\sigma(x)$, in the canonical energy-momentum tensor producing a deviation from the Riemannian geometry. Equations (31) and (32) are simply different ways of expressing the property that the total—that is, acting on both indices—derivative of the tetrad vanishes identically. According to the TSSD map, the next first deformation matrices $\sigma(x) : = (\sigma_b^a)(x)$, contribute to corresponding ingredient part, χ_b^d , of the general deformation tensor:

$$\Omega_b^a = \chi_b^d \overset{\bullet}{\Omega}^a_d = \chi_b^d \overset{\bullet}{\tilde{\Omega}}^\nu_\rho \check{e}_\nu^a \check{e}^\rho_d, \quad \overline{\chi}_d^c = \sigma_e^c \sigma_d^e, \quad \overline{\chi}_e^d \overset{\bullet}{\pi}_b^e = \chi_b^e \overset{\bullet}{\pi}_e^d, \quad (33)$$

or

$$\Omega_\mu^\nu = \chi_\mu^\rho \overset{\bullet}{\tilde{\Omega}}^\nu_\rho, \quad \chi_\mu^\rho = \chi_b^d \check{e}_d^\rho \check{e}^b_\mu. \quad (34)$$

Under a deformation, $\sigma(x)$, in general, the tetrad changes according to

$$\begin{aligned} e_c &= (\sigma_c^d \overset{\bullet}{\pi}_d^a) \check{e}_a = \sigma_c^d \overset{\bullet}{e}_d, & \vartheta^c &= (\sigma_e^c \overset{\bullet}{\pi}_e^b) \check{\vartheta}^b = \sigma_e^c \check{\vartheta}^e, & e_\rho &= \sigma_\rho^\sigma \overset{\bullet}{e}_\sigma, \\ \vartheta^\rho &= \sigma_\sigma^\rho \check{\vartheta}^\sigma, & e_\rho &= \sigma_\rho^c \overset{\bullet}{e}_c, & \vartheta^\rho &= \sigma_c^\rho \check{\vartheta}^c, & \sigma_\rho^c &= \sigma_\rho^\sigma \overset{\bullet}{e}_\sigma^c, & \sigma_c^\rho &= \sigma_\sigma^\rho \overset{\bullet}{e}^\sigma_c, \\ e_c \vartheta^c &= \overline{\chi}_d^c \overset{\bullet}{e}_c \check{\vartheta}^d = \Omega_b^a \check{e}_a \check{\vartheta}^b. \end{aligned} \quad (35)$$

The corresponding second deformation matrices read

$$\gamma_{cd}(x) = \overline{\chi}_{ee'} \dot{\pi}_c^e \dot{\pi}_d^{e'}, \quad \dot{\gamma}_{cd}(\dot{x}) = o_{ab} \dot{\pi}_c^a(\dot{x}) \dot{\pi}_d^b(\dot{x}), \quad (36)$$

where $\overline{\chi}_{ee'} = o_{ab} \sigma_e^a \sigma_{e'}^b$. Under a local tetrad deformation (35), a general spin connection transforms according to

$$\omega'^a_{b\mu} = \sigma_c^a \dot{\omega}^c_{b\mu} \sigma^d_b + \sigma_c^a \partial_\mu \sigma^c_b, \quad (37)$$

such that

$$\overset{(\sigma)}{\omega}^a_{b\mu} : = \omega'^a_{b\mu} = \sigma_c^a \partial_\mu \sigma^c_b, \quad (38)$$

is referred to as the *deformation related frame connection*, which represents the *deformed properties of the frame* only. Then, it follows that the affine connection, Γ , related to (8) and (35) tetrad deformations, transforms through

$$\Gamma^\mu_{\rho\sigma} = e_a^\mu \partial_\sigma e^a_\rho + e_a^\mu \overset{(\pi)}{\omega}^a_{b\sigma} e^b_\rho = \sigma_a^\mu \partial_\sigma \sigma^a_\rho + \sigma_a^\mu \overset{(\sigma)}{\omega}^a_{b\sigma} \sigma^b_\rho, \quad (39)$$

where, according to (35), we have $\sigma_a^\mu \sigma_\mu^b = \delta_a^b$, also the procedure can be inverted $\sigma_a^\mu \sigma_\nu^a = \delta_\nu^\mu$, and that

$$\overset{(\pi)}{\omega}^a_{b\mu} : = \omega^a_{b\mu} = \pi^a_c \check{\omega}^c_{d\mu} \pi^{db} + \pi^a_c \partial_\mu \pi^{cb}, \quad (40)$$

is the spin connection. Taking into account (6), observe that invariants such as the line element, ds^2 , defined on the \mathcal{M}_4 by metric (11) can be alternatively written in a general form of the spacetime or frame objects, respectively, as

$$ds^2 = g_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu = g(e_\mu, e_\nu) \vartheta^\mu \otimes \vartheta^\nu = (\Omega_\mu^\nu \Omega_\rho^\sigma) \check{g}_{\nu\sigma} \check{\vartheta}^\mu \otimes \check{\vartheta}^\rho = o_{ab} \vartheta^a \otimes \vartheta^b = (\Omega_a^c \Omega_b^d) o_{cd} \check{\vartheta}^a \otimes \check{\vartheta}^b = \gamma_{cd} \check{\vartheta}^c \otimes \check{\vartheta}^d. \quad (41)$$

For our convenience, hereinafter the notation, $\{ \overset{(A)}{e}_a, \overset{(A)}{\vartheta}^b \} (A = \pi, \sigma)$, will be used for general linear frames

$$\{ \overset{(A)}{e}_a, \overset{(A)}{\vartheta}^b \} = \{ (\overset{(\pi)}{e}_a, \overset{(\sigma)}{e}_a), (\overset{(\pi)}{\vartheta}^b, \overset{(\sigma)}{\vartheta}^b) \} \equiv \{ (e_a, \dot{e}_a), (\vartheta^b, \dot{\vartheta}^b) \}, \quad (42)$$

where $\overset{(A)}{e}_a \rfloor \overset{(A)}{\vartheta}^b = \delta_a^b$, or in components, $\overset{(A)}{e}_a^\mu \overset{(A)}{e}_\mu^b = \delta_a^b$, also the procedure can be inverted $\overset{(A)}{e}_a^\mu \overset{(A)}{e}_\mu^\sigma = \delta_\rho^\sigma$, provided

$$\overset{(A)}{e}_a^\mu = (\overset{(\pi)}{e}_a^\mu, \overset{(\sigma)}{e}_a^\mu) \equiv (e_a^\mu, \sigma_a^\mu). \quad (43)$$

Hence, the affine connection (39) can be rewritten in the abbreviated form

$$\Gamma^\mu_{\rho\sigma} = \overset{(A)}{e}_a^\mu \partial_\sigma \overset{(A)}{e}_\rho^a + \overset{(A)}{e}_a^\mu \overset{(A)}{\omega}^a_{b\sigma} \overset{(A)}{e}_\rho^b. \quad (44)$$

Since the first deformation matrices $\pi(x)$ and $\sigma(x)$ are arbitrary functions, the transformed general spin connections $\overset{(\pi)}{\omega}(x)$ and $\overset{(\sigma)}{\omega}(x)$, as well as the affine connection (44), are independent of tetrad fields and their derivatives. In what follows, therefore, we will separate the notions of space and connections- the metric-affine formulation of gravity. A metric-affine space $(\mathcal{M}_4, g, \Gamma)$ is defined to have a metric and a linear connection that need not dependent on each other. The new geometrical property of the spacetime are the *nonmetricity* 1-form N_{ab} and the affine *torsion* 2-form T^a representing a translational misfit (for a comprehensive discussion see [8, 9, 12]. These, together with the *curvature* 2-form R_a^b , symbolically can be presented as [5]

$$(N_{ab}, T^a, R_a^b) \sim \mathcal{D}(g_{ab}, \vartheta^a, \Gamma_a^b), \quad (45)$$

where \mathcal{D} is the *covariant exterior derivative*. If the nonmetricity tensor $N_{\lambda\mu\nu} = -\mathcal{D}_\lambda g_{\mu\nu} \equiv -g_{\mu\nu;\lambda}$ does not vanish, the general formula for the affine connection written in the spacetime components is [12]

$$\Gamma^\rho_{\mu\nu} = \overset{\circ}{\Gamma}^\rho_{\mu\nu} + K^\rho_{\mu\nu} - N^\rho_{\mu\nu} + \frac{1}{2} N_{(\mu}^\rho{}_{\nu)}, \quad (46)$$

where $K^\rho_{\mu\nu} := 2Q_{(\mu\nu)}^\rho + Q^\rho_{\mu\nu}$ is the non-Riemann part - the affine *contortion tensor*. The torsion, $Q^\rho_{\mu\nu} = \frac{1}{2}T^\rho_{\mu\nu} = \Gamma^\rho_{[\mu\nu]}$ given with respect to a holonomic frame, $d\vartheta^\rho = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components. In the presence of curvature and torsion, the coupling prescription of a general field carrying an arbitrary representation of the Lorentz group will be

$$\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu - \frac{i}{2}(\omega_\mu^{ab} - K_\mu^{ab})J_{ab}, \quad (47)$$

We may introduce the contortion tensors related to the *deformation-related frame connection* (38) and the spin connection (40):

$${}^{(A)}K^c_{a\nu} = {}^{(A)}\omega^c_{a\nu} + {}^{(A)}\Delta^c_{a\nu}, \quad (48)$$

where

$${}^{(A)}\Delta_{\mu\rho\nu} = e_{\mu a} {}^{(A)}e^a_{[\rho, \nu]} - e_{\rho a} {}^{(A)}e^a_{[\mu, \nu]} - e_{\nu a} {}^{(A)}e^a_{[\mu, \rho]}, \quad (49)$$

is referred to as the the Ricci coefficients of rotation. Both the contortion tensor and spin connection are antisymmetric in their first two indices. The Levi-Civita spin connection

$$\circ {}^{(A)}\omega^\mu_{a\rho} = e^\mu_{a, \rho} = e^\mu_{a, \rho} + \circ \Gamma^\mu_{\nu\rho} e^\nu_a, \quad (50)$$

is related to the Ricci rotation coefficients, with ${}^{(A)}K = 0$, thus,

$${}^{(A)}K^\mu_{a\rho} = {}^{(A)}\omega^\mu_{a\rho} - \circ \omega^\mu_{a\rho}. \quad (51)$$

The relations between the corresponding torsion and contortion tensors read

$${}^{(A)}K^\rho_{\mu\nu} := 2 {}^{(A)}Q^\rho_{(\mu\nu)} + {}^{(A)}Q^\rho_{\mu\nu}, \quad {}^{(A)}Q^\rho_{\mu\nu} = {}^{(A)}K^\rho_{[\mu\nu]}, \quad (52)$$

where

$${}^{(A)}Q^\rho_{\mu\nu} = {}^{(A)}\omega^\rho_{[\mu\nu]} + e^a_{[\mu, \nu]} e^\rho_a. \quad (53)$$

So, the affine connection (44) reads

$$\Gamma^\mu_{\rho\sigma} = \overset{(\sigma)}{\Gamma}^\mu_{(\rho\sigma)} + \overset{(\sigma)}{Q}^\mu_{\rho\sigma} = \overset{(\pi)}{\Gamma}^\mu_{(\rho\sigma)} + \overset{(\pi)}{Q}^\mu_{\rho\sigma}, \quad (54)$$

where $\overset{(\pi)}{\Gamma}^\mu_{(\rho\sigma)} = \overset{(\pi)}{\Gamma}^\mu_{(\rho\sigma)} + 2 \overset{(\pi)}{Q}^\mu_{(\rho\sigma)}$. It is well known that due to the affine character of the connection space [23], one can always add a tensor to a given connection without spoiling the covariance of derivative (47). Let us define then a translation in the connection space. Suppose a point in this space will be a Lorentz connection, $\overset{(\pi)}{\omega}(x) := \overset{(\pi)}{\omega}^{bc}_\mu(x) J_{bc} dx^\mu$, presenting simultaneously curvature and torsion written in the language of differential forms as

$$\overset{(\pi)}{R} = d \overset{(\pi)}{\omega} + \overset{(\pi)}{\omega} \overset{(\pi)}{\omega} \equiv \mathcal{D}_{(\pi)} \overset{(\pi)}{\omega}, \quad \overset{(\pi)}{T} = d e + \overset{(\pi)}{\omega} e \equiv \mathcal{D}_{(\pi)} e, \quad (55)$$

where $\mathcal{D}_{(\pi)}$ denotes the covariant differential in the connection $\overset{(\pi)}{\omega}$. Now, given two connections $\overset{(\sigma)}{\omega}(x)$ and $\overset{(\pi)}{\omega}(x)$, the difference $k = \overset{(\sigma)}{\omega} - \overset{(\pi)}{\omega}$, is also a 1-form assuming values in the Lorentz Lie algebra, but transforming covariantly, whereas its covariant derivative is $\mathcal{D}_{(\pi)} k = dk + \{\overset{(\pi)}{\omega}, k\}$. The effect of adding a covector k to a given connection $\overset{(\pi)}{\omega}$, therefore, is to change its curvature and torsion 2-forms:

$$\overset{(\sigma)}{R} = \overset{(\pi)}{R} + \mathcal{D}_{(\pi)} k + k k, \quad \overset{(\sigma)}{T} = \overset{(\pi)}{T} + k e. \quad (56)$$

Since k^a_{bc} is a Lorentz-valued covector, it is necessarily anti-symmetric in the first two indices. Presenting $k^a_{bc} = \frac{1}{2}k^a_{(bc)} + \frac{1}{2}k^a_{[bc]}$, we may define $k^a_{[bc]} \equiv t^a_{bc}$, such that

$$k^a_{bc} = \frac{1}{2}(t^a_{cb} + t^a_{bc} - t^a_{bc}). \quad (57)$$

Turning to the connection appearing in the covariant derivative (47): $\Omega^{(\pi)}_{bc} \equiv \omega^{(\pi)}_{bc} - K^{(\pi)}_{bc}$, a translation in the connection space with parameter k^a_{bc} corresponds to

$$\Omega^{(\sigma)}_{bc} = \Omega^{(\pi)}_{bc} + k^a_{bc} \equiv \omega^{(\pi)}_{bc} - K^{(\pi)}_{bc} + k^a_{bc}. \quad (58)$$

Since k^a_{bc} has always the form of a contortion tensor (57), the above connection is equivalent to $\Omega^{(\sigma)}_{bc} = \omega^{(\pi)}_{bc} - K^{(\sigma)}_{bc}$, with $K^{(\sigma)}_{bc} = K^{(\pi)}_{bc} - k^a_{bc}$ being another contortion tensor: $K^{(\sigma)\lambda}_{\mu\nu} = K^{(\pi)\lambda}_{\mu\nu} - k^\lambda_{\mu\nu}$. Let us, for example, in (56) choose t^a_{bc} as the torsion of the connection $\omega^{(\pi)}_{bc}$, that is, $t^a_{bc} = T^{(\pi)a}_{bc}$. In this case, $k^a_{bc} = K^{(\pi)a}_{bc}$, and that we are left with the torsionless spin connection of general relativity: $\Omega^{(\sigma)}_{bc} = \overset{\circ}{\omega}_{bc}$. Another example is $t^a_{bc} = T^{(\pi)a}_{bc} - C^{(\pi)a}_{bc}$, when the connection $\omega^{(\pi)}_{bc}$ vanishes, which characterizes teleparallel gravity. In this case, the resulting connection has the form $\omega^{(\sigma)}_{bc} = -\overset{\bullet}{K}_{bc}$, where $\overset{\bullet}{K}_{bc}$ is the contortion tensor written in terms of the Weitzenböck torsion $\overset{\bullet}{T}^a_{bc} = -\overset{\bullet}{C}^a_{bc}$. The particle equation of motion then becomes the force equation of teleparallel gravity. There are actually infinitely many choices for t^a_{bc} , each one defining a different translation in the connection space, and consequently yielding a connection $\omega^{(\pi)}_{bc}$ with different curvature and torsion.

III. TELEPARALLEL GRAVITY

The total covariant derivative of a geometrical object carrying both flat and curvilinear indices is covariant with respect to both diffeomorphism and local Lorentz symmetries. In both, W_4 and U_4 , spaces the total covariant derivative of the vierbein field, e^a_ν , is assumed to vanish

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\nu\mu} e^a_\rho + \omega^a_{b\mu} e^b_\nu = 0, \quad (59)$$

which provides a relation between both connections. Defining the Weitzenböck connection $\overset{\bullet}{\Gamma}^\rho_{\nu\mu} = e_a{}^\rho \partial_\mu e^a_\nu$ ((32)), then (59) leads to

$$\overset{\bullet}{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \omega^a_{\mu b} e_a{}^\rho e^b_\nu = \Gamma^\rho_{\mu\nu} - \omega^\rho_{\mu\nu} = \overset{\circ}{\Gamma}^\rho_{\mu\nu} + K^\rho_{\mu\nu} - \omega^\rho_{\mu\nu}. \quad (60)$$

The anholonomy object C^a and the torsion 2-form T^a , expanded in the anholonomic coframe $\{\vartheta^a\}$, read

$$C^a = \frac{1}{2}C^a_{bc} \vartheta^b \wedge \vartheta^c, \quad T^a = \frac{1}{2}T^a_{bc} \vartheta^b \wedge \vartheta^c, \quad (61)$$

where $T^a_{bc} = C^a_{bc} + \Gamma^a_{bc} - \Gamma^a_{cb}$. Let us now introduce the Weitzenböck torsion $\overset{\bullet}{T}^\rho_{\mu\nu} = \overset{\bullet}{\Gamma}^\rho_{\mu\nu} - \overset{\bullet}{\Gamma}^\rho_{\nu\mu}$, and the Weitzenböck contortion

$$\overset{\bullet}{K}^\rho_{\mu\nu} = \frac{1}{2}(\overset{\bullet}{T}^\rho_{\nu\mu} + \overset{\bullet}{T}^\rho_{\mu\nu} + \overset{\bullet}{T}^\rho_{\mu\nu}). \quad (62)$$

From (59), we then obtain $\overset{\bullet}{T}^\rho_{\mu\nu} = T^\rho_{\mu\nu} - \omega^\rho_{\mu\nu} + \omega^\rho_{\nu\mu}$ and $\overset{\bullet}{K}^\rho_{\mu\nu} = K^\rho_{\mu\nu} + \omega^\rho_{\mu\nu}$. Below, we will concentrate on the specific space, W_4 , of the vanishing affine torsion in the class of frames, $\{e_a\}$: $T^\rho_{\mu\nu} = 0$. Provided, the metricity condition holds: $\overset{\bullet}{N}_{ab} = -\overset{\bullet}{\mathcal{D}} g_{ab} = 0$, and that $\Gamma^\rho_{\mu\nu} = \overset{\circ}{\Gamma}^\rho_{\mu\nu}$, as in the Riemann space. Consequently, $K^\rho_{\mu\nu} = 0$, and $\overset{\bullet}{K}^\rho_{\mu\nu} = \omega^\rho_{\mu\nu}$. Hence, the (59) yields $\overset{\bullet}{\Gamma}^\rho_{\mu\nu} = \overset{\circ}{\Gamma}^\rho_{\mu\nu} + \overset{\bullet}{K}^\rho_{\mu\nu} = \overset{\circ}{\Gamma}^\rho_{\mu\nu} + \omega^\rho_{\mu\nu}$, while, the Weitzenböck covariant derivative of the tetrad field vanishes identically: $\overset{\bullet}{\mathcal{D}}_\nu \overset{\bullet}{e}^a_\mu \equiv \partial_\nu \overset{\bullet}{e}^a_\mu - \overset{\bullet}{\Gamma}^\rho_{\mu\nu} \overset{\bullet}{e}^a_\rho = 0$. This is the so called distant, or absolute parallelism condition. As a consequence of this condition, the corresponding Weitzenböck spin connection also vanishes identically: $\overset{\bullet}{\omega}^c_{a\nu} = \overset{\circ}{\omega}^c_{a\nu} + \overset{\bullet}{K}^c_{a\nu} \equiv 0$. Of course, these relations above are true only in one

specific class of frames. In fact, since $\dot{\omega}^c{}_{a\nu}$ is the Weitzenböck spin connection, if it vanishes in a given frame, it will be different from zero in any other frame related to the first by a local Lorentz transformation. In teleparallel gravity, the coupling of spinor fields with gravitation is a highly controversial subject. However, it seems there are no compelling arguments supporting the choice of the Weitzenböck spin connection $\dot{\omega}^c{}_{a\nu}$ as the spin connection of teleparallel gravity, otherwise several problems are immediately encountered with such coupling prescription. The teleparallel gravity becomes consistent and fully equivalent with GR, even in the presence of spinor fields if we write the minimal coupling prescription as $\dot{\partial}_a \rightarrow \dot{\mathcal{D}}_a = \dot{e}^{\mu}{}_a \dot{\mathcal{D}}_\mu$ with $\dot{\mathcal{D}}_\mu$ the teleparallel Fock-Ivanenko derivative written in the form $\dot{\mathcal{D}}_\mu = \dot{\partial}_\mu - \frac{i}{2} \dot{\Omega}^a{}_{b\mu} J_a{}^b$, where the teleparallel spin connection, $\dot{\Omega}^a{}_{b\mu}$, reads $\dot{\Omega}^a{}_{b\mu} = 0 - \dot{K}^a{}_{b\mu}$. Field equations can be derived from the least action, $\delta \dot{S} = 0$, with the total invariant action of conventional theory of teleparallel gravity.

IV. IN SEARCH OF TSSD-INDUCED DYNAMICAL TORSION

In this section we construct the TSSD- U_4 theory, which considers curvature and torsion as representing independent degrees of freedom. The RC manifold, U_4 , is a particular case of the general metric-affine manifold \mathcal{M}_4 , restricted by the metricity condition $N_{\lambda\mu\nu} = 0$, when a nonsymmetric linear connection is said to be metric compatible. Taking the antisymmetrized derivative of the metric condition gives an identity between the curvature of the spin-connection and the curvature of the Christoffel connection

$$R^{(A)}{}_{\mu\nu}{}^{ab}(\omega) e^{(A)}{}_{\rho b} - R^{(A)}{}_{\rho\mu\nu}(\Gamma) e^{(A)}{}_{\sigma}{}^a = 0, \quad (63)$$

where

$$\begin{aligned} R^{(A)}{}_{\mu\nu}{}^{ab}(\omega) &= \partial_\mu \omega^{(A)}{}_{\nu}{}^{ab} - \partial_\nu \omega^{(A)}{}_{\mu}{}^{ab} + \omega^{(A)}{}_{ac} \omega^{(A)}{}_{\nu}{}^{bc} - \omega^{(A)}{}_{\nu c} \omega^{(A)}{}_{\mu}{}^{ac}, \\ R^{(A)}{}_{\rho\mu\nu}(\Gamma) &= \partial_\mu \Gamma^{(A)}{}_{\nu\rho} - \partial_\nu \Gamma^{(A)}{}_{\mu\rho} - \Gamma^{(A)}{}_{\mu\rho}{}^\lambda \Gamma^{(A)}{}_{\nu\lambda} + \Gamma^{(A)}{}_{\nu\rho}{}^\lambda \Gamma^{(A)}{}_{\mu\lambda}. \end{aligned} \quad (64)$$

Hence, the relations between the scalar curvatures for an U_4 manifold read

$$R^{(A)}(\omega) \equiv e^{(A)}{}_{a\mu} e^{(A)}{}_{b\nu} R^{(A)}{}_{\mu\nu}{}^{ab}(\omega) = R(g, \Gamma) \equiv g^{\rho\nu} R^{(A)}{}_{\rho\mu\nu}(\Gamma). \quad (65)$$

This means that the Lorentz and diffeomorphism invariant scalar curvature, R , becomes either a function of $e^{(A)}{}_{a\mu}$ only, or $g_{\mu\nu}$. Certainly, it can be seen by noting that the Lorentz gauge transformations can be used to fix the six antisymmetric components of $e^{(A)}{}_{a\mu}$ to vanish. Then in both cases diffeomorphism invariance fixes four more components out of the six $g_{\mu\nu}$, with the four components $g_{0\mu}$ being non dynamical, obviously, leaving only two dynamical degrees of freedom. This shows that the equivalence of the vierbein and metric formulations holds.

A. Field equations of dynamical torsion in the tensorial form

According to relations (65), the total *EC action* can be written in the terms of the spin connection, $\omega^{(\pi)}$ and the *deformation-related frame connection*, $\omega^{(\sigma)}$, in the form

$$S = S_g^{(A)}(\omega) + S_m^{(\pi)}(\omega) = -\frac{1}{2\kappa} \int R \sqrt{-g} d\Omega + \int L_m^{(\pi)}(g, \Psi, \nabla \Psi) \sqrt{-g} d\Omega, \quad (66)$$

where $S_g^{(A)}$ ($A = \pi, \sigma$) is the action for the gravitational field written, according to (65), in terms of scalar curvature $R^{(A)}(\omega)$ for a U_4 manifold, while $S_m^{(\pi)}$ is the action for the matter fields, κ is the coupling constant relating to the Newton gravitational constant $\kappa = 8\pi G/c^4$. Action (66) regards the contortion tensor as a variational variable, in addition to the gravitational and matter fields. The gravitational action can be decomposed as

$$S_g^{(A)} = -\frac{1}{2\kappa} \int \overset{\circ}{R} \sqrt{-g} d\Omega + S_Q^{(A)}, \quad (67)$$

where the torsional action reads

$$S_Q^{(A)} = \frac{1}{2\kappa} \int d\Omega \sqrt{-g} L_Q^{(A)} = \frac{1}{2\kappa} \int d\Omega \sqrt{-g} g^{\mu\rho} (2 K^{(A)}{}_{\mu\lambda;\rho} + K^{(A)}{}_{\nu\mu} K^{(A)}{}_{\rho\lambda} - K^{(A)}{}_{\mu\sigma} K^{(A)}{}_{\rho\lambda}), \quad (68)$$

The coupling constant of the spin-torsion is the same of that of the mass-metric distortion field interaction. Partial integration of the terms with covariant derivatives (:) and omitting total derivatives (which do not contribute to the field equations) reduces the action $S_Q^{(A)}$ to

$$S_Q^{(A)} = \frac{1}{2\kappa} \int d\Omega \sqrt{-g} g^{\mu\rho} (K^{(A)}{}^\nu{}_{\mu\nu} K^{(A)}{}^\lambda{}_{\rho\lambda} - K^{(A)}{}^\lambda{}_{\mu\sigma} K^{(A)}{}^\sigma{}_{\rho\lambda}). \quad (69)$$

The corresponding variations can be written as

$$\delta S_Q^{(A)} = \frac{1}{2\kappa} \int [K^{(A)}{}^\nu{}_{\mu\nu} K^{(A)}{}^\lambda{}_{\rho\lambda} - K^{(A)}{}^\lambda{}_{\mu\sigma} K^{(A)}{}^\sigma{}_{\rho\lambda} - \frac{1}{2} g_{\mu\rho} (K^{(A)}{}^{\nu\sigma}{}_\nu K^{(A)}{}^\lambda{}_{\sigma\lambda} - K^{(A)}{}^\lambda{}_\sigma K^{(A)}{}^\sigma{}_{\nu\lambda})] \sqrt{-g} \delta g^{\mu\rho} d\Omega - \frac{1}{\kappa} \int (K^{(A)}{}^{\rho\nu}{}_\mu - K^{(A)}{}^{\lambda\nu}{}_\lambda \delta^\rho_\mu) \sqrt{-g} \delta K^{(A)}{}^\mu{}_{\nu\rho} d\Omega, \quad (70)$$

and

$$\delta S_m^{(\pi)} = \frac{1}{2} \int T_{\mu\rho} \sqrt{-g} \delta g^{\mu\rho} d\Omega + \frac{1}{2} \int S^{(\pi)}{}^\nu{}_{\mu\rho} \sqrt{-g} \delta \omega^{(\pi)}{}^\nu{}_{\mu\rho} d\Omega, \quad (71)$$

where $T_{\mu\rho}$ is the usual dynamical energy-momentum tensor, and $S^{(\pi)}{}^\nu{}_{\mu\rho}$ is the spin tensor. In the metric-affine variational formulation of gravity, the variations $\delta \omega^{(\pi)}{}^{ab}{}_\mu$ are independent of $\delta e_a{}^\mu$ and their derivatives. The dynamical spin density tensor, which is antisymmetric in the Lorentz indices, reads

$$S^{(\pi)}{}_\mu{}^{ab} = 2 \frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta \omega^{(\pi)}{}^\mu{}_{ab}} = 2 \frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta K^{(\pi)}{}^\mu{}_{ab}} = \sqrt{-g} S^{(\pi)}{}_\mu{}^{ab}, \quad (72)$$

where (48) is used. The variation of the action have to be applied by independent variation of the fields $g, \omega^{(\pi)}(x)$ (or equivalently $K^{(\pi)}(x)$) and $\Psi(x), \bar{\Psi}(x)$. In terms of the Euler-Lagrange variations, the least action $\delta S = 0$ gives

$$\begin{aligned} \delta g^{\mu\nu} : \quad \overset{\circ}{G}_{\mu\nu} + \frac{\delta(\sqrt{-g} L_Q^{(A)})}{\delta g^{\mu\nu}} &= -2\kappa \frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta g^{\mu\nu}}; \\ \delta \omega^{(\pi)}{}^\nu{}_{\mu\rho} : \quad \frac{\partial \omega^{(A)}{}^{\mu'\rho'}{}_{\nu}}{\partial \omega^{(\pi)}{}^\nu{}_{\mu\rho}} \frac{\delta}{\delta \omega^{(A)}{}^{\mu'\rho'}{}_{\nu}} (\sqrt{-g} L_Q^{(A)}) &= -\frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta \omega^{(\pi)}{}^\nu{}_{\mu\rho}}; \\ \delta \Psi : \quad \frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta \Psi} &= 0; \quad \delta \bar{\Psi} : \quad \frac{\delta(\sqrt{-g} L_m^{(\pi)})}{\delta \bar{\Psi}} = 0, \end{aligned} \quad (73)$$

where $\overset{\circ}{G}_{\mu\nu}$ is the Einstein tensor

$$\overset{\circ}{G}_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - \frac{1}{2} \overset{\circ}{R} g_{\mu\nu}, \quad (74)$$

provided we have

$$\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_Q^{(A)})}{\delta g^{\mu\nu}} = \kappa U_{\mu\nu}, \quad (75)$$

where

$$U_{\mu\nu} = \frac{1}{\kappa} [K^{(A)}{}^\nu{}_{\mu\nu} K^{(A)}{}^\lambda{}_{\rho\lambda} - K^{(A)}{}^\lambda{}_{\mu\sigma} K^{(A)}{}^\sigma{}_{\rho\lambda} - \frac{1}{2} g_{\mu\rho} (K^{(A)}{}^{\nu\sigma}{}_\nu K^{(A)}{}^\lambda{}_{\sigma\lambda} - K^{(A)}{}^\lambda{}_\sigma K^{(A)}{}^\sigma{}_{\nu\lambda})], \quad (76)$$

or

$$\begin{aligned} U_{\mu\nu} &= \frac{1}{\kappa} [-(Q^{(A)}{}^\lambda{}_{\mu\nu} + 2 Q^{(A)}{}_{(\mu\nu)}{}^\lambda)(Q^{(A)}{}^\nu{}_{\rho\lambda} + 2 Q^{(A)}{}_{(\rho\lambda)}{}^\nu) + 4 Q^{(A)}{}_\mu Q^{(A)}{}_\rho + \frac{1}{2} g_{\mu\rho} (Q^{(A)}{}^{\sigma\nu\lambda} + \\ &2 Q^{(A)}{}_{(\nu\lambda)}{}^\sigma)(Q^{(A)}{}_{\lambda\nu\sigma} + 2 Q^{(A)}{}_{(\nu\sigma)}{}^\lambda) - 2g_{\mu\rho} Q^{(A)}{}^\nu{}_\nu Q^{(A)}{}_\nu]. \end{aligned} \quad (77)$$

By virtue of relation (75), we may recast the first equation in (73) into the *first EC equation*

$$\overset{\circ}{G}_{\mu\nu} = \kappa \theta^{(A)}{}_{\mu\nu}, \quad (78)$$

where including the spin contributions directly into the energy-momentum tensor, we introduce the canonical energy-momentum tensor

$$\overset{(A)}{\theta}_{\mu\nu} = T_{\mu\nu} + \overset{(A)}{U}_{\mu\nu}. \quad (79)$$

For variations $\delta \overset{(\pi)}{K}^{\nu}_{\mu\rho}$ (or equivalently $\overset{(\pi)}{\omega}^{\nu}_{\mu\rho}$), the $\delta S = 0$ gives the *second EC equation*

$$\frac{\partial \overset{(A)}{\omega}^{\mu'\rho'}}{\partial \overset{(\pi)}{\omega}^{\nu\mu\rho}} \overset{(A)}{\mathcal{T}}^{\nu}_{\mu'\rho'} = -\frac{1}{2} \mathfrak{a} \overset{(\pi)}{S}^{\nu}_{\mu\rho}, \quad (80)$$

where the *modified torsion* reads

$$\overset{(A)}{\mathcal{T}}^{\nu}_{\mu\rho} = \frac{1}{2\sqrt{-g}} \frac{\delta(\sqrt{-g} L_Q^{(A)})}{\delta \overset{(A)}{\omega}^{\nu\mu\rho}} = \overset{(A)}{Q}^{\nu}_{\mu\rho} + \delta^{\nu}_{\mu} \overset{(A)}{Q}_{\rho} - \delta^{\nu}_{\rho} \overset{(A)}{Q}_{\mu}. \quad (81)$$

Thus, the equations of the standard EC theory can be recovered for $A = \pi$:

$$\overset{\circ}{G}_{\mu\nu} = \mathfrak{a} \overset{(\pi)}{\theta}_{\mu\nu}, \quad \overset{(\pi)}{\mathcal{T}}^{\nu}_{\mu\rho} = -\frac{1}{2} \mathfrak{a} \overset{(\pi)}{S}^{\nu}_{\mu\rho}, \quad (82)$$

in which the equation defining torsion is the algebraic type, such that torsion at a given point in spacetime does not vanish only if there is matter at this point, represented in the Lagrangian density by a function which depends on torsion. Unlike the metric, which is related to matter through a differential field equation, torsion does not propagate. Combining (77), (81) and (82) gives

$$\begin{aligned} \overset{(\pi)}{U}_{\mu\nu} = \mathfrak{a} \left(- \overset{(\pi)}{S}^{\rho}_{\mu} [\overset{(\pi)}{S}^{\lambda}_{\nu\rho}] - \frac{1}{2} \overset{(\pi)}{S}^{\rho\lambda}_{\mu} \overset{(\pi)}{S}_{\nu\rho\lambda} + \right. \\ \left. \frac{1}{4} \overset{(\pi)}{S}^{\rho\lambda}_{\mu} \overset{(\pi)}{S}_{\rho\lambda\nu} + \frac{1}{8} g_{\mu\nu} (-4 \overset{(\pi)}{S}^{\lambda}_{\rho} [\overset{(\pi)}{S}^{\rho\tau}_{\lambda}] + \overset{(\pi)}{S}^{\rho\lambda\tau} \overset{(\pi)}{S}_{\rho\lambda\tau}) \right). \end{aligned} \quad (83)$$

However, equations (82) can be equivalently replaced by the set of *modified EC equations* for $A = \sigma$:

$$\overset{\circ}{G}_{\mu\nu} = \mathfrak{a} \overset{(\sigma)}{\theta}_{\mu\nu}, \quad \Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\overset{(\sigma)}{\mathcal{T}}) \overset{(\sigma)}{\mathcal{T}}^{\nu'}_{\mu'\rho'} = -\frac{1}{2} \mathfrak{a} \overset{(\pi)}{S}^{\nu}_{\mu\rho}, \quad (84)$$

where

$$\frac{\partial \overset{(\sigma)}{\omega}^{\mu'\rho'}}{\partial \overset{(\pi)}{\omega}^{\nu\mu\rho}} \overset{(\sigma)}{\mathcal{T}}^{\nu}_{\mu'\rho'} = \Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\overset{(\sigma)}{\mathcal{T}}) \equiv \Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\pi(x), \sigma(x)), \quad (85)$$

in which the torsion $\overset{(\sigma)}{\mathcal{T}}^{\nu}_{\mu\rho}$ is *dynamical* if only $\Theta^{\mu'\rho'\nu}_{\nu'\mu\rho}(\pi(x), \sigma(x)) \neq \delta^{\mu'}_{\mu} \delta^{\rho'}_{\rho} \delta^{\nu}_{\nu}$. According to (79), it is spin that generates a nonsymmetric part in the canonical energy-momentum tensor and then, produces a deviation from the Riemann geometry. The variation of $S_m^{(\pi)}$ (68) with respect to the metric-compatible affine connection in the metric-affine variational formulation of gravity is equivalent to the variation with respect to the torsion (or contortion) tensor.

Consequently, the dynamical spin density $\overset{(\pi)}{s}^{\mu}_{ab}$ is identical with

$$\overset{(\pi)}{\Sigma}^{\mu}_{ab} = \frac{\partial(\sqrt{-g} L_m)^{(\pi)}}{\partial \Psi_{,\mu}} \overset{(\pi)}{S}_{ab} \Psi, \quad (86)$$

referred to as the *canonical spin density*. The canonical tensor $e \overset{(A)}{\theta}_{\mu\nu} = \overset{(A)}{\tau}_{\mu\nu} = e^a_{\mu\nu} \overset{(A)}{\tau}^a_{\mu}$ is generally not symmetric, whereas the canonical energy-momentum density is identical with the dynamical tetrad energy-momentum density $e \overset{(A)}{\theta}^a_{\mu} = \overset{(A)}{\tau}^a_{\mu}$, where $e := \det[e^{\mu}_a] = \sqrt{-g}$. The relation between the tetrad dynamical energy-momentum tensor and the metric dynamical energy-momentum tensor for matter fields is $\overset{(A)}{\theta}_{(\mu\nu)} = T_{\mu\nu}$. The Belinfante-Rosenfeld relation, between the dynamical metric and dynamical tetrad (canonical) energy-momentum tensors, can be written as (see e.g. [12]):

$$\overset{(A)}{\theta}_{\mu\nu} - T_{\mu\nu} = \frac{1}{2} \nabla^*_{\nu} (\overset{(\pi)}{S}^{\nu}_{\mu\rho} - \overset{(\pi)}{S}^{\nu}_{\rho\mu} + \overset{(\pi)}{S}^{\nu}_{\mu\rho}) = \overset{(A)}{U}_{\mu\nu}, \quad (87)$$

where $\nabla_\mu^* = \nabla_\mu - 2 \overset{(\pi)}{Q}_\mu$ is the *modified covariant derivative*. The conservation law for the spin density results from antisymmetrizing the Belinfante-Rosenfeld relation with respect to the indices μ, ρ :

$$\overset{(\pi)}{S}_{\mu\nu}{}^\rho{}_{;\rho} = \tau_{\mu\nu} - \tau_{\nu\mu} + 2 \overset{(\pi)}{Q}_\rho \overset{(\pi)}{S}_{\mu\nu}{}^\rho \quad (88)$$

$$\text{or } \frac{1}{2} \nabla_\rho^* \overset{(\pi)}{S}_{\mu\nu}{}^\rho = \overset{(A)}{\theta}_{[\mu\nu]}.$$

B. TSSD- U_4 theory in the language of differential forms

In this subsection we re-derive the field equations of the TSSD- U_4 theory by using the exterior calculus. The fields have to be expressed in terms of differential forms in order to build the total Lagrangian 4-form as the appropriate integrand of the action. Let $\overset{(A)}{\omega}{}^{ab} = \overset{(A)}{\omega}_\mu{}^{ab} \wedge dx^\mu$ be the 1-forms of corresponding connections assuming values in the Lorentz Lie algebra. The action for gravitational field can be written in the form

$$S_g = \overset{\circ}{S} + S_Q = -\frac{1}{4\kappa} \int \star \overset{\circ}{R} + S_Q, \quad (89)$$

where \star denotes the Hodge dual. This is a C^∞ -linear map $\star : \Omega^p \rightarrow \Omega^{n-p}$, which acts on the wedge product monomials of the basis 1-forms as $\star(\vartheta^{a_1 \dots a_p}) = \varepsilon^{a_1 \dots a_p} e_{a_{p+1} \dots a_n}$. Here e_{a_i} ($i = p+1, \dots, n$) are understood as the down indexed 1-forms $e_{a_i} = o_{a_i b} \vartheta^b$ and $\varepsilon^{a_1 \dots a_n}$ is the total antisymmetric pseudo-tensor. According to (65), the relations between the Ricci scalars read

$$\overset{\circ}{R} \equiv \overset{\circ}{R}_{cd} \wedge \overset{\bullet}{\vartheta}{}^c \wedge \overset{\bullet}{\vartheta}{}^d = \overset{\circ}{R}_{cd} \wedge \vartheta^c \wedge \vartheta^d. \quad (90)$$

Consider a phenomenological action of the spin-torsion interaction, S_Q , such that the variation of the connection 1-form $\overset{(A)}{\omega}{}^{ab}$ yields

$$\delta S_Q = \frac{1}{\kappa} \int \star \overset{(A)}{\mathcal{T}}_{ab} \wedge \delta \overset{(A)}{\omega}{}^{ab}, \quad (91)$$

where

$$\star \overset{(A)}{\mathcal{T}}_{ab} = \frac{1}{2} \star (\overset{(A)}{Q}_a \wedge \overset{(A)}{e}_b) = \overset{(A)}{Q}{}^c \wedge \overset{(A)}{\vartheta}{}^d \varepsilon_{cdab} = \frac{1}{2} \overset{(A)}{Q}{}^c{}_{\mu\nu} \wedge \overset{(A)}{e}{}^d{}_\alpha \varepsilon_{abcd} \overset{(A)}{\vartheta}{}^{\mu\nu\alpha}, \quad (92)$$

here we used the abbreviated notations for the wedge product monomials, $\overset{(A)}{\vartheta}{}^{\mu\nu\alpha\dots} = \overset{(A)}{\vartheta}{}^\mu \wedge \overset{(A)}{\vartheta}{}^\nu \wedge \overset{(A)}{\vartheta}{}^\alpha \wedge \dots$, defined on the U_4 space, and that

$$\overset{(A)}{Q}{}^a = \overset{(A)}{D} \overset{(A)}{\vartheta}{}^a = d \overset{(A)}{\vartheta}{}^a + \overset{(A)}{\omega}{}^a{}_b \wedge \overset{(A)}{\vartheta}{}^b. \quad (93)$$

The variation of the action describing the macroscopic matter sources $S_m^{(\pi)}$ with respect to the coframe ϑ^a , and connection 1-form $\overset{(\pi)}{\omega}{}^{ab}$ reads

$$\delta S_m = \int \delta L_m = \int (-\star \overset{(A)}{\theta}{}_a \wedge \delta \overset{(A)}{\vartheta}{}^a + \frac{1}{2} \star \overset{(\pi)}{\Sigma}_{ab} \wedge \delta \overset{(\pi)}{\omega}{}^{ab}), \quad (94)$$

where $\star \overset{(A)}{\theta}{}_a$ is the dual 3-form relating to the canonical energy-momentum tensor, $\overset{(A)}{\theta}{}^\mu{}_a$, by

$$\star \overset{(A)}{\theta}{}_a = \frac{1}{3!} \overset{(A)}{\theta}{}^\mu{}_a \varepsilon_{\mu\nu\alpha\beta} \overset{(A)}{\vartheta}{}^{\nu\alpha\beta}. \quad (95)$$

and $\star \overset{(\pi)}{\Sigma}_{ab} = -\star \overset{(\pi)}{\Sigma}_{ba}$ is the dual 3-form corresponding to the canonical spin tensor, which is identical with the dynamical spin tensor $\overset{(\pi)}{S}_{abc}$, namely

$$\star \overset{(\pi)}{\Sigma}_{ab} = \overset{(\pi)}{S}{}^\mu{}_{ab} \varepsilon_{\mu\nu\alpha\beta} \overset{(\pi)}{\vartheta}{}^{\nu\alpha\beta}. \quad (96)$$

The integral

$$\overset{\circ}{S} = -\frac{1}{4\mathfrak{a}} \int \star \overset{\circ}{R} = -\frac{1}{4\mathfrak{a}} \int \star \overset{\circ}{R}_{cd} \wedge \overset{(A)}{\vartheta}^c \wedge \overset{(A)}{\vartheta}^d, \quad (97)$$

is the usual Einstein action, written in the language of the exterior forms. Actually, writing explicitly the holonomic indices, we have

$$\overset{\circ}{S} = -\frac{1}{8\mathfrak{a}} \int \overset{\circ}{R}_{\mu\nu}{}^{ab} \overset{(A)}{e}^c{}_\alpha \overset{(A)}{e}^d{}_\beta \varepsilon_{abcd} \overset{(A)}{\vartheta}^{\mu\nu\alpha\beta} = -\frac{1}{8\mathfrak{a}} \int \overset{\circ}{R}_{\mu\nu}{}^{ab} \varepsilon_{ab\alpha\beta} \varepsilon^{\mu\nu\alpha\beta} d\Omega. \quad (98)$$

Using the relations

$$\varepsilon_{ab\alpha\beta} \varepsilon^{\mu\nu\alpha\beta} = -2e \overset{(A)}{e}^{\mu\nu}{}_{ab}, \quad \overset{(A)}{e}^{\mu\nu}{}_{ab} = \begin{vmatrix} \overset{(A)}{e}^\mu{}_a & \overset{(A)}{e}^\nu{}_a \\ \overset{(A)}{e}^\mu{}_b & \overset{(A)}{e}^\nu{}_b \end{vmatrix}, \quad (99)$$

we have

$$\overset{\circ}{S} = -\frac{1}{2\mathfrak{a}} \int \overset{\circ}{R}_{\mu\nu}{}^{ab} \overset{(A)}{e}^\mu{}_\alpha [\overset{(A)}{e}^\nu{}_\beta] e d\Omega = -\frac{1}{2\mathfrak{a}} \int \overset{\circ}{R} \sqrt{-g} d\Omega. \quad (100)$$

Also, one may readily verify that

$$\delta S_Q = \frac{1}{\mathfrak{a}} \int \overset{(A)}{\mathcal{T}}_{\mu\nu}{}^\beta \delta \overset{(A)}{\omega}_\beta{}^{\mu\nu}. \quad (101)$$

Certainly,

$$\begin{aligned} \delta S_Q &= \frac{1}{2\mathfrak{a}} \int \overset{(A)}{Q}^c{}_{\mu\nu} \overset{(A)}{e}^\alpha{}_d \delta \overset{(A)}{\omega}_\beta{}^{ab} \varepsilon_{cdab} \overset{(A)}{\vartheta}^{\mu\nu\alpha\beta} = \\ &= -\frac{1}{2\mathfrak{a}} \int \overset{(A)}{Q}^c{}_{\mu\nu} \delta \overset{(A)}{\omega}_\beta{}^{ab} \varepsilon_{\alpha cab} \varepsilon^{\alpha\mu\nu\beta} d\Omega. \end{aligned} \quad (102)$$

Using the relations $\varepsilon_{\alpha cab} \varepsilon^{\alpha\mu\nu\beta} = -e \overset{(A)}{e}^{\mu\nu\beta}{}_{cab}$, where

$$\overset{(A)}{e}^{\mu\nu\beta}{}_{cab} = \begin{vmatrix} \overset{(A)}{e}^\mu{}_c & \overset{(A)}{e}^\nu{}_c & \overset{(A)}{e}^\beta{}_c \\ \overset{(A)}{e}^\mu{}_a & \overset{(A)}{e}^\nu{}_a & \overset{(A)}{e}^\beta{}_a \\ \overset{(A)}{e}^\mu{}_b & \overset{(A)}{e}^\nu{}_b & \overset{(A)}{e}^\beta{}_b \end{vmatrix}, \quad (103)$$

we obtain

$$\frac{1}{2} \overset{(A)}{Q}^c{}_{\mu\nu} \overset{(A)}{e}^{\mu\nu\beta}{}_{cab} = \overset{(A)}{e}^\mu{}_a \overset{(A)}{e}^\nu{}_b \overset{(A)}{\mathcal{T}}_{\mu\nu}{}^\beta. \quad (104)$$

and (102) gives (101). The variation of the total action, given by the sum of the gravitational field action and the matter action, with respect to the e^a , $\overset{(\pi)}{\omega}^{ab}$, and Ψ , gives

$$\begin{aligned} 1) \quad & \frac{1}{2} \overset{\circ}{R}_{cd} \wedge \overset{(A)}{\vartheta}^c = \mathfrak{a} \overset{(A)}{\theta}_d, \quad 2) \quad \frac{\partial \overset{(A)}{\omega}_{a'b'}}{\partial \overset{(\pi)}{\omega}_{ab}} \wedge \star \overset{(A)}{\mathcal{T}}_{a'b'} = -\frac{1}{2} \mathfrak{a} \star \overset{(\pi)}{\Sigma}_{ab}, \\ 3) \quad & \frac{\delta L_m^{(\pi)}}{\delta \Psi} = 0, \quad \frac{\delta L_m^{(\pi)}}{\delta \Psi} = 0, \end{aligned} \quad (105)$$

In the tensor language then the first equation in (105) coincides with the tensorial equation (78). To prove this, we may recast it into the form

$$\frac{1}{4} \overset{\circ}{R}_{\mu\nu}{}^{ab} \overset{(A)}{e}^c{}_\alpha \varepsilon_{abcd} \varepsilon^{\mu\nu\alpha\beta} = \frac{1}{3!} \mathfrak{a} \overset{(A)}{\theta}^a{}_d \varepsilon_{a\mu\nu\alpha} \varepsilon^{\mu\nu\alpha\beta} \overset{(A)}{\vartheta}^{\nu\alpha\beta}, \quad (106)$$

such that

$$-\frac{\mathfrak{e}}{4} \overset{\circ}{R}_{\mu\nu}{}^{ab} \overset{(A)}{e}^{\mu\nu\beta}{}_{abc} = \mathfrak{a} \overset{(A)}{\theta}^a{}_c \overset{(A)}{e}^\beta{}_a. \quad (107)$$

Making use of the relation

$$-\circ R_{\mu\nu}^{(A)ab} e_{abc}^{(A)\mu\nu\beta} = -2 \circ R^{(A)(A)} e_c^\beta + 4 \circ R_c^\beta, \quad (108)$$

finally, gives

$$\circ R_c^\beta - \frac{1}{2} e_c^\beta \circ R = \mathfrak{e} \theta_\alpha^{(A)\beta}. \quad (109)$$

We may evaluate the second equation in (105) as

$$\frac{\partial \omega_{\beta}^{(A)cd}}{\partial \omega_{\beta'}^{(\pi)c'd'}} \wedge \frac{1}{2} Q^a \wedge e^b \varepsilon_{abcd} = -\frac{1}{2} \mathfrak{e} \star \Sigma_{cd}^{(\pi)}, \quad (110)$$

and that

$$\frac{\partial \omega_{\beta}^{(A)cd}}{\partial \omega_{\beta'}^{(\pi)c'd'}} \frac{1}{2} Q^a{}_{\mu\nu} e_{\alpha}^{(A)b} \varepsilon_{abcd} \varepsilon^{\mu\nu\alpha\beta} = -\frac{1}{2} \mathfrak{e} S_{c'd'}^{(\pi)a} \varepsilon_{a\mu\nu\alpha} \varepsilon^{\mu\nu\alpha\beta'}, \quad (111)$$

so

$$\frac{\partial \omega_{\beta}^{(A)cd}}{\partial \omega_{\beta'}^{(\pi)c'd'}} \frac{e}{2} Q^a{}_{\mu\nu} e^{(A)\mu\nu\beta}_{acd} = -\frac{e}{2} \mathfrak{e} S_{c'd'}^{(\pi)a} e_a^{\beta'} = -\frac{e}{2} \mathfrak{e} S^{\beta'}_{c'd'}. \quad (112)$$

Taking into account relation (104), we then obtain

$$\frac{\partial \omega_{\beta}^{(A)cd}}{\partial \omega_{\beta'}^{(\pi)c'd'}} \mathcal{T}_{cd}^{(A)\beta} = -\frac{1}{2} \mathfrak{e} S^{\beta'}_{c'd'}, \quad (113)$$

which concises with the equation (80).

C. Short-range spin-spin interaction

In this subsection we derive the equations of the short-range propagating torsion, which is of fundamental importance from a view point of microphysics. This, together with the torsion waves, may contribute a new special polarized effect in the current experiments of a verification of gravitational spin-torsion interaction. These experiments include neutron interferometry, neutron spin rotation induced by torsion in vacuum, anomalous spin-dependent forces with a polarized mass torsion pendulum, space-based searches for spin in gravity, the neutrino oscillations, etc., see e.g. [24]. For instance, in the case of torsion, the fact that neutrino oscillations are possible also if neutrinos are massless is very important because, in general, it is thought that if one finds neutrino oscillations the neutrinos must have a mass different from 0. This would be an interesting topic not discussed in this paper. A remarkable feature of the present theoretical work is describing a propagating torsion (84). Furthermore, this in a natural way can be made a short-range propagating torsion. Actually, from the equations (80) and (81), we see that it is the spin $S^{(\pi)}$ and spacetime deformations $\pi(x)$ and $\sigma(x)$ that define the torsion $Q^{(A)}$:

$$Q^{(A)\nu}{}_{\mu\rho} = \mathcal{T}^{\nu}{}_{\mu\rho}^{(A)} + \frac{1}{2} \delta_{\mu}^{\nu} \mathcal{T}^{\lambda}{}_{\rho\lambda}^{(A)} - \frac{1}{2} \delta_{\rho}^{\nu} \mathcal{T}^{\lambda}{}_{\mu\lambda}^{(A)}, \quad (114)$$

which through definitions (77), (79) and the field equation (78), in turn, defines Einstein's field tensor $\overset{\circ}{G}$. A generic spacetime deformation, $\pi(x)$, consists of two ingredient deformations $(\overset{\bullet}{\pi}(x), \sigma(x))$ of the orthonormal frame. Whereas, when the deformation matrix $\overset{\bullet}{\pi}(x)$ implies a peculiar condition (24), the choice of the $\sigma(x)$ is not fixed yet. This allows us to impose a physical constraint upon the spacetime deformation $\sigma(x)$:

$$\Theta_{\nu'\mu\rho}^{\mu'\rho'\nu}(\pi(x), \sigma(x)) = (\Box + M_{\mathcal{T}}^2) \mathcal{T}^{(\sigma)\nu}{}_{\mu\rho} (\mathcal{T}^{-1})_{\nu'}^{\mu'\rho'}, \quad (115)$$

where \square is a generalization of the d'Alembertian operator for covariant derivatives defined on the RC manifold, U_4 . Then, the set of *modified EC equations* (84) reduces to

$$\overset{\circ}{G}_{\mu\nu} = \mathfrak{a} \overset{(\sigma)}{\theta}_{\mu\nu}, \quad (\square + M_{\mathcal{T}}^2) \overset{(\sigma)}{\mathcal{T}}^{\nu}{}_{\mu\rho} = -\frac{1}{2} \mathfrak{a} \overset{(\pi)}{S}^{\nu}{}_{\mu\rho}, \quad (116)$$

which describe the short-range propagating torsion and spin-spin interaction. Actually, at large distances $r > \lambda_{\mathcal{T}} \equiv \frac{\hbar}{M_{\mathcal{T}} c}$ (Compton length), torsion vanishes $\overset{(\sigma)}{\mathcal{T}}(r) = 0$. To carry through this theory in full generality, for example, we may explicitly write the torsionic equation for the Dirac spinor matter source coupled to the metric and to the torsion. Both are contained implicitly in the connection $\overset{(\pi)}{\omega}{}^{ba}{}_{\mu}$. The Dirac spinor field defined in the TSSD- U_4 theory coincides with a conventional formalism of the spinor field defined on the RC spacetime [8, 12]. The Lagrangian of spinor field written for any frame of reference is

$$e L_{\psi}^{(\pi)} = \frac{ie}{2} (\bar{\psi} \overset{(\pi)}{g}{}^{\mu}{}_{\nu} \psi_{,\mu} - \bar{\psi}_{,\mu} \overset{(\pi)}{g}{}^{\mu}{}_{\nu} \psi) - \frac{ie}{2} \bar{\psi} \{ \overset{(\pi)}{g}{}^{\mu}{}_{\nu} \overset{(\pi)}{\Gamma}_{\mu} \} \psi_{,\mu} - m e \bar{\psi} \psi, \quad (117)$$

where γ^a are Dirac matrices, and $\overset{(A)}{g}{}^{\mu}{}_{\nu} = \overset{(A)}{e}{}^{\mu}{}_a \gamma^a$. The spinor connection $\overset{(\pi)}{\Gamma}_{\mu}$ is given, up to the addition of an arbitrary vector multiple of I , by the *Fock-Ivanenko coefficients*

$$\overset{(\pi)}{\Gamma}_{\mu} = -\frac{1}{4} \overset{(\pi)}{\omega}{}_{ab\mu} \gamma^a \gamma^b = -\frac{1}{2} \overset{(\pi)}{\omega}{}_{ab\mu} S^{ab} = -\frac{1}{8} e^{\nu}{}_{c;\mu} [\overset{(\pi)}{g}{}^{\nu}{}_{\nu}, \gamma^c] = \frac{1}{8} [\overset{(\pi)}{g}{}^{\nu}{}_{;\mu}, \overset{(\pi)}{g}{}^{\nu}{}_{\nu}], \quad (118)$$

with $S^{ab} = \frac{1}{2} \gamma^{[a} \gamma^{b]} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a)$ - the spinor representation. Therefore, in the absence of other sources of torsion, the RC manifold, U_4 , with a Dirac field will be characterized by the Lagrangian

$$e L_{\psi}^{(\pi)} = \frac{ie}{2} (\bar{\psi} \overset{(\pi)}{g}{}^{\mu}{}_{\nu} \psi_{,\mu} - \bar{\psi}_{,\mu} \overset{(\pi)}{g}{}^{\mu}{}_{\nu} \psi) + \frac{ie}{8} \overset{(\pi)}{\omega}{}_{ab\mu} \bar{\psi} \{ \overset{(\pi)}{g}{}^{\mu}{}_{\nu}, \gamma^a \gamma^b \} \psi_{,\mu} - m e \bar{\psi} \psi, \quad (119)$$

Using the identity $\{ \overset{(\pi)}{g}{}^{\mu}{}_{\nu}, \overset{(\pi)}{g}{}^{\nu}{}_{\rho} \overset{(\pi)}{g}{}^{\rho}{}_{\sigma} \} = 2 \overset{(\pi)}{g}{}^{[\mu}{}_{\nu} \overset{(\pi)}{g}{}^{\nu}{}_{\rho} \overset{(\pi)}{g}{}^{\rho}{}_{\sigma]}$, the totally antisymmetric spin corresponding to the Lagrangian density (119) is

$$\overset{(\pi)}{S}{}^{\mu\nu\rho} = \overset{(\pi)}{S}{}^{[\mu\nu\rho]} = \frac{i}{2} \bar{\psi} \overset{(\pi)}{g}{}^{[\mu}{}_{\nu} \overset{(\pi)}{g}{}^{\nu}{}_{\rho} \overset{(\pi)}{g}{}^{\rho}{}_{\sigma}] \psi. \quad (120)$$

Consequently, we may recast the torsionic equation into the form

$$(\square + M_{\mathcal{T}}^2) \overset{(\sigma)}{\mathcal{T}}^{\nu}{}_{\mu\rho} = -\frac{i}{4} \mathfrak{a} \bar{\psi} \overset{(\pi)}{g}{}^{[\mu}{}_{\nu} \overset{(\pi)}{g}{}^{\nu}{}_{\rho} \overset{(\pi)}{g}{}^{\rho}{}_{\sigma}] \psi, \quad (121)$$

where ψ implies the *Heisenberg-Ivanenko* nonlinear equation which can be derived from the Lagrangian (119) by means of the standard calculation:

$$i \overset{(\pi)}{g}{}^{\rho}{}_{\sigma} \psi_{;\rho} - \frac{3\mathfrak{a}}{8} (\bar{\psi} \overset{(\pi)}{g}{}^{\rho}{}_{\sigma} \gamma^5 \psi) \overset{(\pi)}{g}{}^{\rho}{}_{\sigma} \gamma^5 \psi = m \psi, \quad (122)$$

This is the Dirac equation written in the so-called second-order formalism, in which the contortion tensor is given explicitly in terms of the spin sources. In the limit when we neglect the usual Riemannian terms depending on the metric and the curvature ($\partial_{;\mu} \rightarrow \partial_{\mu}$), as we are interested only in the spin-torsion interaction, we then have

$$\overset{(\sigma)}{\mathcal{T}}^{\nu}{}_{\mu\rho}(x) = \frac{\mathfrak{a}}{2} \int G_F(x, x') \overset{(\pi)}{S}{}^{\nu}{}_{\mu\rho}(x') d^4 x' \quad (123)$$

where the Feynman propagator reads

$$G_F(x, x') = \begin{pmatrix} -\frac{1}{4\pi} \delta(s) + \frac{M_{\mathcal{T}}}{8\pi\sqrt{s}} H_1^{(1)}(M_{\mathcal{T}}\sqrt{s}) & \text{if } s \geq 0 \\ -\frac{iM_{\mathcal{T}}}{4\pi^2\sqrt{-s}} K_1(M_{\mathcal{T}}\sqrt{-s}) & \text{if } s < 0, \end{pmatrix} \quad (124)$$

provided $s = (x - x')^2$, $H_1^{(1)}$ is the Hankel function of first kind and K_1 is a modified Bessel function. A detailed analysis and calculations on the more general MAG theory with dynamical torsion in context of TSSD formulation of post-Riemannian geometry will be presented in another paper to follow at a later date.

V. CONCLUDING REMARKS

We show how the curvature and torsion, which are properties of a connection of geometry under consideration, will come into being? The theoretical significance resides in constructing the theory of TSSD as a guiding principle. In this, we have to separate from the very outset the case of teleparallel gravity, in which the torsion is a propagating field, from the case of Einstein-Cartan (EC) theory, in which it is not. This motivates our specific choice of the general spacetime deformation $\pi(x)$ of the orthonormal frame, to be consisted of two ingredient deformations ($\dot{\pi}(x)$, $\sigma(x)$). We choose the first deformation matrix $\dot{\pi}(x)$ in such a way ((24) or (30)) that the deformed connection is set as the Weitzenböck connection. We consider then the different affine connections (54), with different curvature and torsion, associated with the deformation-related frame connection (38) and a general spin connection (40). They are independent of tetrad fields and their derivatives. Therefore, we separate the notions of space and connections, namely we take a spacetime simply as a manifold, and affine connections as additional structures- the metric-affine formulation of gravity. Defining a translation in the connection space, we construct the TSSD-versions of the theory of teleparallel gravity and the EC theory. It is remarkable that the equations of the standard EC theory, in which the equation defining torsion is the algebraic type and, in fact, no propagation of torsion is allowed, can be equivalently replaced by the set of *modified EC equations* in which the torsion, in general, is dynamical. Furthermore, we assume a special physical ansatz (115) for the spacetime deformation $\sigma(x)$, which yields the short-range propagating spin-spin interaction. This, together with the torsion waves, may contribute a new special polarized effect in the current experiments of a verification of gravitational spin-torsion interaction.

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